Minor Research Project entitled
"Modeling inventory problems in supply chain through differential equations"

Name of PI : Dr. Ajay S Gor
File No : 47-1003/09 (WRO) Dt 22/09/2009

Executive Summary

In this project, an integrated supplier-buyer inventory model is studied when the units in inventory are subject to deterioration at a constant rate, the market demand is quadratic and sensitive to retailer price and the supplier offers a trade credit. The trade credit policy under assumption is "two-part" strategy viz "Net credit". That is retailer has a choice between cash discount and trade credit. "Net credit" means if the buyer pays within M_1, the buyer receives a cash discount; otherwise, the full account must be settled before M_2 where $M_2 > M_1 \geq 0$. The goal is to determine the optimal pricing, ordering, shipping and payment policy to maximize the joint profit per unit time. An algorithm is given to obtain optimal solution. The numerical example is given to validate the proposed model and to arrive at managerial insights.

Keywords: Integrated inventory model, Net credit, Price sensitive quadratic demand, Deterioration

1. Introduction:

It is customary prevailing amongst the supplier to offer trade credit to boost the demand of the product and reduce on-hand stock level. During offered credit period, the buyer's capital investment ties up in stock, that results to a reduction in the buyer's holding cost of finance. During this allowable settlement period, the buyer may earn interest on the generated revenue.The demand of fashionable merchandise, seasonal paddy grains, air tickets during christmas vacation etc. increases exponentially with time. Here an attempt is made to derive an integrated inventory model when units in inventory system are subject to constant rate of deterioration. The demand is assumed to be price and time dependent. The model is derived under the assumption that the supplier offers the buyer a cash discount if payment is made before a specified period, and if the buyer fails to pay within the allowable trade credit, the full payment must be paid before the due date of the trade credit. The optimal payment time, retail price, ordering policy and number of shipments from supplier to the buyer is derived to maximize the joint total profit. An algorithm is given to determine optimal solution. Numerical example is given to support the proposed model.

2. Assumptions and Notations:

The model is developed using following assumptions and notations:

1. The supply chain of single supplier - single buyer is considered.
2. The integrated inventory system deals with a single item.
3. Shortages are not allowed. Lead-time is zero.
4. The carrying charge fraction excluding interest charges for the supplier is I_s and for the buyer is I_b.
5. To increase cash inflow and reduce the risk of cash flow shortage, the supplier offers a discount, β $(0 < \beta < 1)$ off the purchase price, if the buyer settles the account within time M_1, otherwise, full due payment is to be made within permissible delay period M_2, where $M_2 > M_1 \geq 0$.
6. The supplier's unit manufacturing cost is $\$C_s$ and unit sale price is \$v. The buyer's unit sale price is \$P. P is a decision variable. The relation between these costs is $P > v > (1 - \beta)\, v > C_s$.
7. While offering a trade credit to the buyer, the supplier agrees to give up an immediate cash inflow. Thus, the supplier incurs a capital opportunity cost at rate I_{sp} during the time between delivery and payment of the product.
8. During period $[M_1, M_2]$, a cash flexibility rate f_{sc} is used to quantize the advantage of early cash income for the supplier.
9. During the allowable credit period, the buyer generates the revenue by selling the product. This sales revenue is deposited in an interest bearing account at the rate I_{be}. At the end of this period, the supplier charges to the buyer at the rate I_{bc} on the unsold stock.
10. The market demand rate for the item is increasing function of time and decreasing function of the retail price. Take $R(P, t) = a(1 + bt + ct^2)P^{-\eta}$, where $a > 0$ is constant demand, $0 < b, c < 1$ are rates of change of demand and $\eta > 1$ is a price elasticity mark-up.
11. The capacity utilization γ is defined as the ratio of the market demand rate to production rate. $\gamma < 1$ is known constant.
12. The buyer's cycle time T is a decision variable.
13. The buyer's ordering cost per order is A_b.
14. During the production period, the supplier produces in batches of size nQ (where n is an n integer and decision variable). The set-up cost for a batch is A_s. Once the first Q – units are produced, the supplier transport to the buyer and then makes continuous delivery at every T- time units until the supplier's inventory falls to zero.
15. The units on hand are subject to constant rate of deterioration (say) θ, $0 < \theta < 1$. There is no repair or replacement of the deteriorated units during the cycle time.

3. Mathematical Model:

3.1 Supplier's total profit per unit time:

During each production run, the supplier manufacturers in batches of size nQ where $Q = aP^{-\eta}[(1+bT+cT^2)\frac{e^{\theta T}}{\theta} - (b+2cT)\frac{e^{\theta T}}{\theta^2} + 2c\frac{e^{\theta T}}{\theta^3} - \frac{1}{\theta} + \frac{b}{\theta^2} - \frac{2c}{\theta^3}]$ and incurs a batch setup cost A_s. The supplier's setup cost per unit time is $\frac{A_s}{nT}$. With the unit production cost C_s, the inventory carrying charge fraction I_s excluding interest charges and the capital opportunity cost per dollar per unit time I_{sp}, using Joglekar (1988), the supplier's inventory holding cost per unit time is given by

$$\frac{1}{T}\ C_s(I_s + I_{sp})[(n\text{-}1)(1\text{-}\gamma) + \gamma]\}\ *$$

$$aP^{-\eta}[(1+bT+cT^2)\frac{e^{\theta T}}{\theta^2} - (b+2cT)\frac{e^{\theta T}}{\theta^3} + 2c\frac{e^{\theta T}}{\theta^4} - \frac{1}{\theta}(T + \frac{b}{2}T^2 + \frac{c}{3}T^3) - \frac{1}{\theta^2} + \frac{b}{\theta^3} - \frac{2c}{\theta^4}]$$

For each unit of item, the supplier charges $(1\text{-}k_j\beta)v$ if the buyer pays the payment at M_j, where j = 1, 2; $k_1 = 1$ and $k_2 = 0$**.** The opportunity cost per unit time for trade credit is

$$(1\text{-}k_j\beta)vI_{sp}aP^{-\eta}M_j[1 + \frac{bT}{2} + \frac{cT^2}{3}]$$

If the buyer pays at M_1, during $(M_2 - M_1)$,the supplier can utilize the revenue $(1\text{-}\beta)v$ to reduce a cash flow crisis. With a cash flexibility rate f_{sc}, the gain from early payment is

$$(1\text{-}\beta)vf_{sc}[1 + \frac{bT}{2} + \frac{cT^2}{3}](M_2 - M_1)$$

Hence , the supplier's total profit per unit time is sales revenue minus total cost which consists of the manufacturing cost, set-up cost, inventory holding cost and opportunity cost for offering trade credit and plus the early payment is given by

$$TSP_j(n) = \frac{(1-k_j\beta)vQ}{T} - \frac{C_sQ}{T} - \frac{A_s}{nT}$$

$$-\frac{1}{T} * C_s(I_s + I_{sp})[(n\text{-}1)(1\text{-}\gamma) + \gamma]\} *$$

$$aP^{-\eta}[(1+bT+cT^2)\frac{e^{\theta T}}{\theta^2} - (b+2cT)\frac{e^{\theta T}}{\theta^3} + 2c\frac{e^{\theta T}}{\theta^4} - \frac{1}{\theta}(T + \frac{b}{2}T^2 + \frac{c}{3}T^3) - \frac{1}{\theta^2} + \frac{b}{\theta^3} - \frac{2c}{\theta^4}]$$

$$-\ (1\text{-}k_j\beta)vI_{sp}aP^{-\eta}M_j[1 + \frac{bT}{2} + \frac{cT^2}{3}]$$

$$+ k_j(1-\beta)vf_{sc}aP^{-\eta}[1+\frac{bT}{2}+\frac{cT^2}{3}](M_2-M_1)$$

$$j = 1, 2;\ k_1 = 1 \text{ and } k_2 = 0. \qquad (1)$$

3.2 Buyer's total profit per unit time:

The ordering cost for replenishment of Q-units is A_b. So the ordering cost per unit time is $\frac{A_b}{T}$. The buyer's unit purchase cost is $\frac{(1-k_j\beta)vQ}{T}$ and inventory holding cost is

$$\frac{1}{T}*(1-k_j\beta)vI_b\, aP^{-\eta}[(1+bT+cT^2)\frac{e^{\theta T}}{\theta^2}-(b+2cT)\frac{e^{\theta T}}{\theta^3}+2c\frac{e^{\theta T}}{\theta^4}-\frac{1}{\theta}(T+\frac{b}{2}T^2+\frac{c}{3}T^3)-\frac{1}{\theta^2}+\frac{b}{\theta^3}-\frac{2c}{\theta^4}]$$

Depending on the payment time, following two cases may arise (i) $T < M_j$ and (ii) $T \geq M_j$, j = 1, 2. These two cases are shown in Fig. 1.

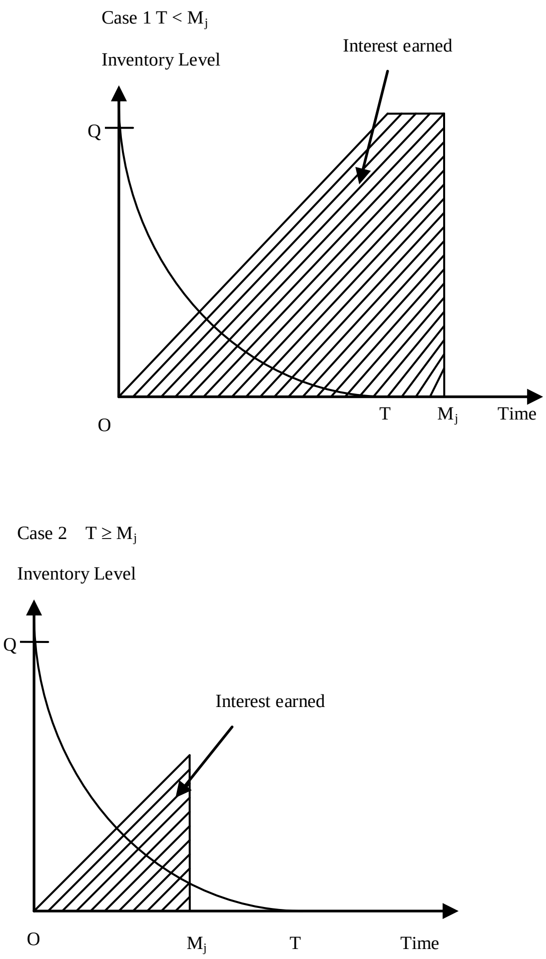

Fig. 1 Inventory status for the buyer under trade credit

Case 1: $T < M_j$, $j = 1, 2$.

Here, the buyer's stock depletes to zero before allowable trade credit. So, buyer has no opportunity cost for unsold stock. The buyer can earn interest on the generated revenue at the rate I_{be} and is given by

$$\frac{1}{T}[PI_{be}\int_0^T R(P,t)tdt + PI_{be}\, R(P,T)T(M_j - T)]$$

$$= PI_{be}aP^{-\eta}[(1+bT+CT^2)M_j - \frac{T}{2} - \frac{2bT^2}{3} - \frac{3cT^3}{4}]$$

Hence, buyer's total profit per unit time is

$$TBP_{j1} = \frac{PQ}{T} - \frac{(1-k_j\beta)vQ}{T} - \frac{A_b}{T} -$$

$$\frac{1}{T} * (1\text{-}k_j\beta)vI_b\, aP^{-\eta}[(1+bT+cT^2)\frac{e^{\theta T}}{\theta^2} - (b+2cT)\frac{e^{\theta T}}{\theta^3} + 2c\frac{e^{\theta T}}{\theta^4} - \frac{1}{\theta}(T + \frac{b}{2}T^2 + \frac{c}{3}T^3) - \frac{1}{\theta^2} + \frac{b}{\theta^3} - \frac{2c}{\theta^4}]$$

$$+ aI_{be}P^{-\eta+1}[(1+bT+CT^2)M_j - \frac{T}{2} - \frac{2bT^2}{3} - \frac{3cT^3}{4}],\ j = 1, 2 \tag{2}$$

Case 2 : $M_j \le T$, $j = 1, 2$.

In this scenario, he buyer's delay payment time ends on or before the cycle time. During $[0, M_j]$, the buyer can earn interest at the rate I_{be} on the generated sales. The interest earned per unit time is

$$\frac{PI_{be}}{T}\int_0^{M_j} R(P,t)tdt = \frac{aI_{be}P^{-\eta+1}}{T}[\frac{M_j^2}{2} + \frac{bM_j^3}{3} + \frac{cM_j^4}{4}]$$

During $[M_j, T]$, the buyer will have to pay interest at the rate I_{bc} on the unsold stock. Hence, the interest payable per unit time by the buyer is

$$\frac{(1-k_j\beta)vI_{bc}}{T}\int_{M_j}^{T} I(t)dt =$$

$$\frac{(1-k_j\beta)vI_{bc}}{T}\, aP^{-\eta}\{(1+bT+cT^2)\frac{e^{\theta(T-M_j)}}{\theta^2} - (b+2cT)\frac{e^{\theta(T-M_j)}}{\theta^3} + 2c\frac{e^{\theta(T-M_j)}}{\theta^4} -$$

$$-\frac{1}{\theta}[(T + \frac{b}{2}T^2 + \frac{c}{3}T^3) - (M_j + \frac{b}{2}M_j^2 + \frac{c}{3}M_j^3)] - \frac{1}{\theta^2}(1 + bM_j + cM_j^2) + \frac{1}{\theta^3}(b + 2cM_j) - \frac{2c}{\theta^4}\}$$

The buyer's procures Q - units. So his purchase cost per unit time is $\frac{(1-k_j\beta)vQ}{T}$ and he generates revenue as $\frac{PQ}{T}$. Therefore, total profit of buyer per unit time is

$$TBP_{j2}(P,T) = \frac{PQ}{T} - \frac{(1-k_j\beta)vQ}{T} - \frac{A_b}{T} -$$

$$\frac{1}{T}*(1\text{-}k_j\beta)vI_b\ aP^{-\eta}[(1+bT+cT^2)\frac{e^{\theta T}}{\theta^2}-(b+2cT)\frac{e^{\theta T}}{\theta^3}+2c\frac{e^{\theta T}}{\theta^4}-\frac{1}{\theta}(T+\frac{b}{2}T^2+\frac{c}{3}T^3)-\frac{1}{\theta^2}+\frac{b}{\theta^3}-\frac{2c}{\theta^4}]$$

$$+\frac{aI_{be}p^{-\eta+1}}{T}[\frac{M_j^2}{2}+\frac{bM_j^3}{3}+\frac{cM_j^4}{4}]\ -$$

$$\frac{(1-k_j\beta)vI_{bc}}{T}aP^{-\eta}\{(1+bT+cT^2)\frac{e^{\theta(T-M_j)}}{\theta^2}-(b+2cT)\frac{e^{\theta(T-M_j)}}{\theta^3}+2c\frac{e^{\theta(T-M_j)}}{\theta^4}-$$

$$-\frac{1}{\theta}[(T+\frac{b}{2}T^2+\frac{c}{3}T^3)-(M_j+\frac{b}{2}M_j^2+\frac{c}{3}M_j^3)]-\frac{1}{\theta^2}(1+bM_j+cM_j^2)+\frac{1}{\theta^3}(b+2cM_j)-\frac{2c}{\theta^4}\}$$

$$, j = 1, 2 \quad (3)$$

The total profit for the buyer per time unit is

$$\mathrm{TBP_j(P,T)} = \begin{cases} TBP_{j1}(P,T) & \text{If } T < M_j; \\ TBP_{j2}(P,T) & \text{If } T \geq M_j \end{cases} \qquad j = 1,2 \qquad (4)$$

3.3 The joint total profit per unit time:

The supplier-buyer joint profit per unit time is given by

$$\pi_j\ (n,P,T) = \begin{cases} \pi_{j1}(n,P,T) & \text{if } \ T < M_j; \ \ j = 1,2 \\ \pi_{j2}(n,P,T) & \text{if } \ T \geq M_j \end{cases} \qquad (5)$$

where π_{j1} (n, P, T) = $TSP_j(n)$ + TBP_{j1} (P, T)

and π_{j2} (n, P, T) = $TSP_j(n)$ + TBP_{j2} (P, T), j = 1,2

The aim is to determine the optimal values of discrete variable n and continues variables P and T, which maximizes π_j (n, P, T), j = 1, 2.

4. Computational Algorithm:

To maximize total joint profit, perform following steps:

Step 1 : Assign parametric values to all model parameters.

Step 2 : Initialize n = 1.

Step 3 : Solve $\frac{\partial \pi_j}{\partial P} = 0$ and $\frac{\partial \pi_j}{\partial T} = 0$ for j = 1, 2

Step 4 : Increment n by 1.

Step 5 : Continue steps 2 and 3 till , we get

$$\pi_j(\text{n-1, P(n-1), T(n-1)}) \geq \pi_j\ (\text{n, P(n), T(n)}) \ \leq \pi_j\ (\text{n+1, P(n+1), T(n+1)})$$

Step 6 : Stop

Knowing the optimal solution (n, P, T), the optimal purchase quantity Q=R(P,T) per replenishment for the buyer can be obtained.

5. Numerical Example:

Consider the integrated supplier-buyer inventory system with the following numerical values:

a = 90,000, b=0.02, c=0.05, γ = 0.9, η = 1.25, Cs = \$2/unit, v = \$4.5/unit, A_s= \$1000/setup, A_b = \$300/order, I_s = 0.05/unit/annum, I_b = 0.08/unit/year, I_{sp} =0.09/\$/year, I_{bc} = 0.16/\$/year, I_{be} = 18/\$/year and f_{sc} = 0.17/ \$/year ,θ=0.05 and a credit term as "2/10 net 30" means M_1, = 10 days, M_2 = 30 days and β= 2% is offered by the supplier.

For 9 - replenishments, the buyer’s selling price is \$10.27 and cycle time T_{12} = 48.83 days which gives maximum joint profit \$401333, supplier’s profit as \$117649 and buyer’s profit as \$283684. The optimum order quantity per order is 6581 units. Optimal payment time is 10 days in 2/10 net 30 policy. The graph of n versus total joint profit (fig 2) , P versus total joint profit (fig 3) and 3-D plot of total joint profit for n=9 reveals the concavity of the profit function (fig. 4).

Table 1 Performances of supply chain for various credit terms

M_1 (days)	M_2 (days)	Optimal payment time (days)	n	P (\$)	T (days)	R(P,T) (units)	Q (units)	Profit % Buyer	Supplier	Joint	Profit gain Buyer	Supplier	Joint
0	0	-	9	10.54	T=50.98	47552	6652	282041	116675	398716	-	-	-
0	30	30	9	10.21	$\boldsymbol{T_2}$ = 49.23	49484	6685	283496	117610	401106	0.52	0.80	0.60
10		10	9	10.27	$\boldsymbol{T_{12}}$ = 48.83	49127	6581	283684	117649	401333	0.58	0.83	0.66
20		20	10	10.30	T_{12} = 43.20	48950	5801	285490	115761	401251	1.22	-0.78	0.63
0	60	60	16	10.31	T_1 = 29.62	48799	3964	290707	116778	407485	3.07	0.09	2.20
10		60	16	10.31	T_{21} = 29.62	48799	3964	290707	116778	407485	3.07	0.09	2.20
20		60	16	10.31	T_{21} = 29.62	48799	3964	290707	116778	407485	3.07	0.09	2.20
0	90	90	14	10.32	T_1 = 30.43	13206	4069	298286	115020	413306	5.75	-1.42	3.66
10		90	14	10.32	T_{21} = 30.43	13206	4069	298286	115020	413306	5.75	-1.42	3.66
20		90	14	10.32	T_{21} = 30.43	48746	4069	298286	115020	413306	5.75	-1.42	3.66

In table 1, the effects of variations in M_1 and M_2 on decision variables and objective functions are shown. The profit gain is compared with no credit period. The profit gain is defined as

[(Profit with trade credit - Profit without trade credit)/ Profit without trade credit] * 100%

It is observed that profit gain is positive when supplier - buyer agrees to work jointly. That is total profit for supply chain as a whole under two level trade credits is beneficial than the individual decision. When allowable payment time is 90 – days, supplier's profit gain is negative. It is also seen that buyer gets attracted to pay early under the scheme of "2/10 net 30" and maximizes his profit.

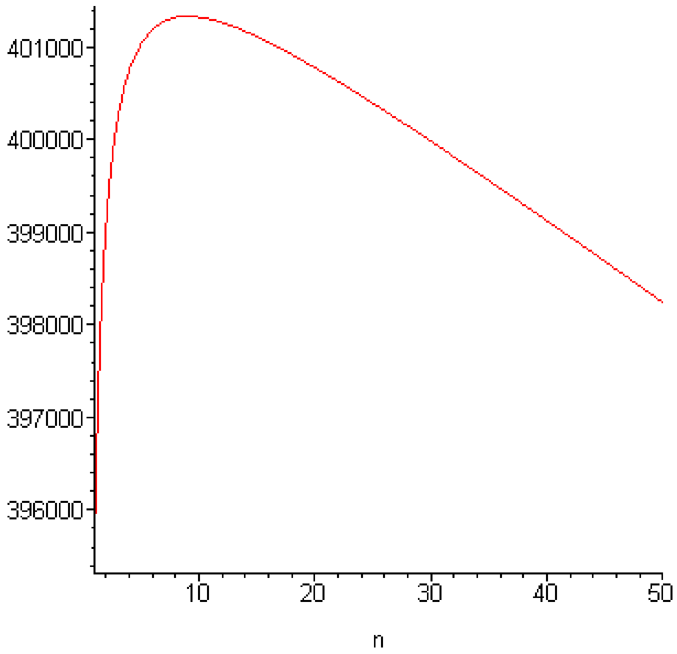

Fig. 2 Concavity of joint total profit w.r.t. n

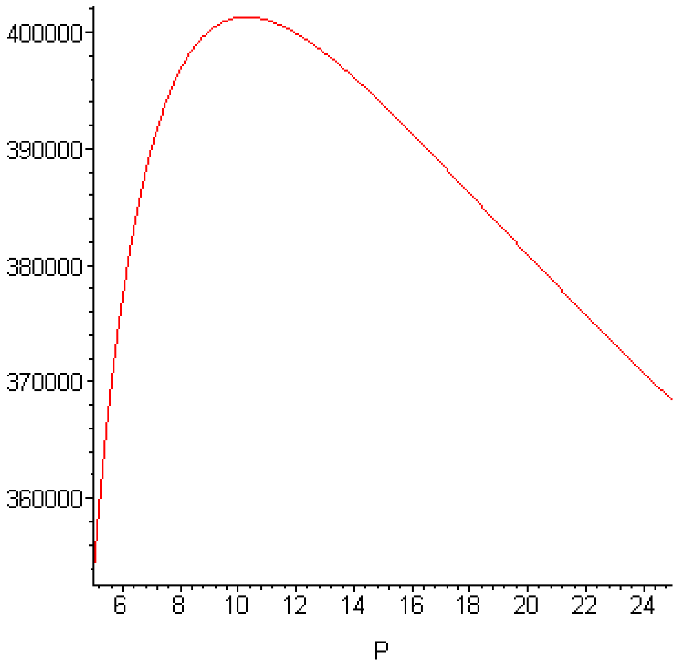

Fig. 3 Concavity of joint total profit w.r.t. P

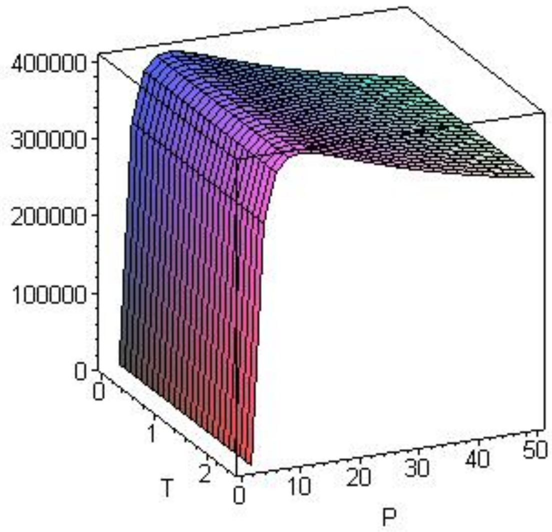

Fig. 4 **Concavity of joint total profit for n=9**

Table 2 Optimum solutions of supply chain under different strategy

Strategy	Credit term	payment time (days)	*n*	*P* ($)	*T* (days)	*R(P,T)* (units)	*Q* (units)	*nQ*	Profit ($) Buyer	Supplier	Joint
Independent	Cash on delivery	0	8	23.63	T=106.89	17120	4205	33640	327861	42215	370076
	trade credit **2/10** net 30	10	8	22.71	$\boldsymbol{T_{12}}$ = 102.35	18752	4223	33784	330375	42974	373349
Joint	Cash on delivery	0	9	10.54	T=50.98	47552	6652	59868	282041	116675	398716
	trade credit **2/10** net 30	10	9	10.27	$\boldsymbol{T_{12}}$ = 48.83	49127	6581	59229	283684	117649	401333
	Adjusted								355138	46195	401333

In table 2, independent and joint strategies are studied. The optimal solutions of "cash on delivery" (Putting $M_1 = 0$, $M_2 = 0$ and $\beta = 0$) and "2/10 net 30" under individual and joint decision are exhibited. It is observed that under both decisions, offering trade credit to the buyer lowers retail price and increases demand. However, when the supplier and buyer take decision independently, irrespective of allowable trade credit or not, the buyer's retail price is double. The procurement order is smaller when sole decision is taken and thereby lowering joint profit. It is also seen that buyer's profit decreases and that of supplier increases, which forces the buyer to take decision independently. Therefore, in order to establish beneficial strategy for both the players of the supply chain, we apply a simple compensation method suggested by Goyal (1976). Readjust π (n, P, T) and obtain

$$\text{Buyer's profit} = \pi\,(n, P, T) \;*\; \frac{TBP(P,T)}{[TBP(P,T) + TSP(n)]}$$

$$= 401333 * \frac{330375}{373349} = 355138$$

$$\text{Supplier's profit} = \pi\,(n, P, T) \;*\; \frac{TSP(n)}{TBP(P,T) + TSP(n)}$$

$$= 401333 * \frac{42974}{373349} = 46195$$

The adjusted profit is shown in the last row of table 2.

6. Conclusions:

In this work , a joint supplier-buyer inventory model is derived when demand is price and time sensitive. The units in inventory are subject to deterioration at a constant rate. It is assumed that the supplier offers two payment options namely, β / M_1 net M_2. The joint profit is maximized with respect to the best payment time, the sale price, the purchase quantity and number of shipment per production to be transported from the supplier to the buyer. It is observed that a two-level credit scenario increases joint profits of the supply chain. It is established that supplier can share additional profit by joint venture to encourage that buyer to cooperate.

Acknowledgements: This research work is supported in part by UGC project scheme # 47 – 1003/09 (WRO).